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On form-preserving point transformations of partial differential equations

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Abstract. New identities are presented relating arbitrary order partial derivatives of $u(x,t)$ and $u'(x',t')$ for the general point transformation $x' = P(x,t,u)$, $t' = Q(x,t,u)$, $u' = R(x,t,u)$. These identities are used to study the nature of those point transformations which preserve the general form of a wide class of $1+1$ partial differential equations. These results facilitate the search for point symmetries, both discrete and continuous (Lie), and assist the search for point transformations which reduce equations to canonical, but similar, form. A simple test for the existence of hodograph-type transformations between equations of similar form is given.

1. Introduction

Probably the most useful point transformations of partial differential equations (PDEs) are those which form a continuous (Lie) group of transformations, each member of which leaves an equation invariant. Symmetries of this equation are then revealed, perhaps suggesting links with equations studied in a different context, perhaps enabling new solutions to be found directly or via similarity reductions.

The classical method of finding these transformations is first to find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations. However, this method may well overlook discrete symmetries such as simple reflection or hodograph transformations. Also infinitesimal transformations are not appropriate for directly linking a PDE with an equation of a different form. This is useful, for example, when converting equations to a canonical form on which an established theory can be called.

An example of a discrete symmetry is given by Kingston and Sophocleous [1] who found that the reciprocal transformation (double application gives the identity transformation) $x' = x/t, t' = 1/t, u' = -(ut - x)$ leaves the Burger-type equation $u_t + uu_x + (f(t) - f(1/t))u_{xx} = 0$ invariant, a symmetry additional to the Lie point symmetries obtained from the classical approach. Further, this reciprocal transformation, modified to $x' = \mathrm{i}x/(\alpha t), t' = 1/(\alpha^2 t), u' = -\mathrm{i}\alpha(ut - x)$, provides the missing link, postulated by Doyle and Englefield [2], between $u_t + uu_x + \mathrm{e}^{1/\alpha t}u_{xx} = 0$ and $u'_{t'} + u'u'_{x'} + \mathrm{e}^{\alpha t'}u'_{x'x'} = 0$ which they had shown to possess the same Lie algebras.

Ordinary differential equations furnish other examples of when Lie symmetries obtained by the classical method only provide a subgroup of the full point symmetry group. Olver [3, p184], cites the example $u_{xx} = xu + \tan(u_x)$ which has no continuous symmetry yet possesses the discrete reflection symmetry $u \to -u$. A more interesting example

(Reid *et al* [4], Englefield [5]) is $u_{xx} = u_x/x + 4u^2/x^3$ which admits the scaling symmetry group $x' = \alpha x, u' = \alpha u$, in addition to a cyclic group of order 4 generated by $x' = x^{\mathrm{i}}, u' = -ux^{\mathrm{i}-1} - \frac{1}{4}x^{\mathrm{i}}$.

Discrete point transformations between equations of similar form have also been used by Chen [6] to find Bäcklund transformations. In the AKNS scheme the linear representation of nonlinear equations has eigenfunction $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}$, say, which when eliminated can produce soliton equations, for example the KdV equation $u_t + u_{xxx} + 12uu_x = 0$. The linear representation also shows that $v = \Psi_1/\Psi_2$ satisfies two Riccati equations, one for each independent variable x and t. If u (see above equation) is eliminated from these Riccati equations, the resulting equation for v is $v_t + v_{xxx} - 24v^2v_x + 24kvv_x = 0$. Chen showed how the simple reflection $v \to -v$, which leads to a PDE of a similar form ($k \to -k$) can be used to derive the classic Bäcklund transformation for the KdV equation. However, this approach is probably more for interest than for efficiency of method.

Changing PDEs to canonical forms is another application of discrete point transformations. A model of nonlinear waves in a weakly inhomogeneous plasma was given by Zakharov [7] as the variable coefficient cubic Schrödinger equation

$$\mathrm{i}u'_{t'} + u'_{x'x'} + 2u'^2u'^* - 2\alpha x'u' = 0.$$

Chen and Liu [8] used the point transformations

$$x' = x - 2\alpha t^2 \qquad t' = t \qquad u' = u\exp[-2\mathrm{i}\alpha xt + \tfrac{8}{3}\mathrm{i}\alpha^2t^3]$$

to convert the above equation to the cubic Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + 2u^2u^* = 0.$$

A second example is the conversion of the cylindrically symmetric nonlinear diffusion equation

$$u'_{t'} = \frac{1}{x'}(x'u'^{-1}u'_{x'})_{x'}$$

to the one-dimensional equation

$$u_t = (u^{-1}u_x)_x.$$

King [9] achieved this with the point transformation $x' = \mathrm{e}^x, t' = t, u' = \mathrm{e}^{-2x}u$ and went on, [10], to generalize and exploit this.

The above reasons and examples show that there is merit in studying point transformations directly in finite form with the ultimate dual goals of finding the complete set of point transformation symmetries of PDEs and discovering new links between different equations.

The aim of this paper is first to present results concerning the relation of the transformed partial derivatives to the original partial derivatives and secondly to exploit these results to reduce the general range of point transformations connecting PDEs belonging to restricted classes of equations.

In section 2 we explain the notation and summarize the basic theory on which the work herein is based.

Relationships between partial derivatives are considerably more cumbersome than the corresponding relationships for infinitesimal transformations which themselves expand rapidly with increasing order. However several manageable results are presented in section 3 as lemmas 3.1–3.3 and corollaries 3.1–3.3 (to lemma 3.1). The proofs are given in appendix A. Section 3 finishes with the tentative unproven conjecture 3.1.

The results of section 3 help us achieve the second aim of the paper which is to discover the nature of point transformations connecting PDEs belonging to given classes of equations. Thus, in section 4 we first look at PDEs with one partial derivative of $u(x, t)$ of any order, possibly mixed, related to lower-order derivatives of u, u itself, and x and t. That brief study is summarized by theorem 4.1, which is followed by a simple test for the existence of hodograph-type transformations between equations of certain types. Subsequently, we consider three classes of equations. If H represents a function of x, t, u and derivatives of u we discuss evolution equations $u_t = H$ and also the two classes $u_{tt} = H$ and $u_{xt} = H$. The results are summarized in theorems 4.2a, b, 4.3a–c and 4.4a–c.

2. Point transformations: Notation and basic theory

We consider the point transformation

$$x' = P(x, t, u) \qquad t' = Q(x, t, u) \qquad u' = R(x, t, u) \tag{2.1}$$

relating $x, t, u(x, t)$ and $x', t', u'(x', t')$, and assume that this is non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P, Q, R)}{\partial(x, t, u)} \neq 0 \tag{2.2}$$

and also that

$$\delta = \frac{\partial(P(x, t, u(x, t)), Q(x, t, u(x, t)))}{\partial(x, t)} \neq 0. \tag{2.3}$$

In (2.3) P and Q are expressed as functions of x and t whereas in (2.2) P, Q and R are to be regarded as functions of the independent variables x, t, u.

The derivatives of $u(x, t)$ and $u'(x', t')$ will be denoted by

$$u_{ij} = \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \qquad u'_{ij} = \frac{\partial^{i+j} u'}{\partial x'^i \partial t'^j}. \tag{2.4}$$

If Ψ is a function of x, t, u and the derivatives of u, the total derivatives of Ψ with respect to x and t will be denoted by

$$\Psi_X = \Psi_x + \sum\sum u_{i+1j} \frac{\partial \Psi}{\partial u_{ij}} \tag{2.5}$$

$$\Psi_T = \Psi_t + \sum\sum u_{ij+1} \frac{\partial \Psi}{\partial u_{ij}} \tag{2.6}$$

where the double summations are to be taken over the values of i and j which cover all derivatives u_{ij} occurring in Ψ.

With this notation δ may be expressed as

$$\begin{aligned} \delta = \frac{\partial(P, Q)}{\partial(X, T)} &= P_X Q_T - P_T Q_X \\ &= -u_{10}(P_t Q_u - P_u Q_t) - u_{01}(P_u Q_x - P_x Q_u) + (P_x Q_t - P_t Q_x). \end{aligned} \tag{2.7}$$

Also, under the point transformation (2.1),

$$\begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix} = \begin{pmatrix} P_X & P_T \\ Q_X & Q_T \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix} \qquad \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix} = \frac{1}{\delta} \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix} \tag{2.8}$$

and

$$\mathrm{d}\Psi = \Psi_X \mathrm{d}x + \Psi_T \mathrm{d}t = \frac{1}{\delta} \begin{pmatrix} \Psi_X & \Psi_T \end{pmatrix} \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix}. \tag{2.9}$$

Hence, taking $\Psi = u'_{ij-1}, u'_{i-1j}$ respectively, gives

$$u'_{ij} = \delta^{-1}(P_X(u'_{ij-1})_T - P_T(u'_{ij-1})_X) \qquad j \geqslant 1, i \geqslant 0 \tag{2.10}$$

$$u'_{ij} = \delta^{-1}(Q_T(u'_{i-1j})_X - Q_X(u'_{i-1j})_T) \qquad i \geqslant 1, j \geqslant 0. \tag{2.11}$$

Also,

$$u'_{00} = u' = R. \tag{2.12}$$

Equations (2.10)–(2.12) furnish recurrence relations which enable u'_{ij} to be expressed in terms of x, t, u and the derivatives of u for any $i \geqslant 0$, $j \geqslant 0$. The factor δ^{-1} makes the expressions for u'_{ij} grow with i and j in a very cumbersome manner.

In the case of infinitesimal Lie point transformations in which

$$\begin{aligned} P(x,t,u) &= x + \epsilon P^*(x,t,u) + \mathrm{O}(\epsilon^2) \\ Q(x,t,u) &= t + \epsilon Q^*(x,t,u) + \mathrm{O}(\epsilon^2) \\ R(x,t,u) &= u + \epsilon R^*(x,t,u) + \mathrm{O}(\epsilon^2) \end{aligned} \tag{2.13}$$

the forms of J and δ in (2.2) and (2.3) simplify to

$$J = 1 + \epsilon(P^*_x + Q^*_t + R^*_u) \tag{2.14}$$

$$\delta = 1 + \epsilon(P^*_X + Q^*_T) \tag{2.15}$$

to the first order of ϵ. The recurrence relations corresponding to (2.10)–(2.12) are

$$u'_{ij} = (u'_{ij-1})_T - \epsilon[P^*_T(u'_{ij-1})_X + Q^*_T(u'_{ij-1})_T] \qquad j \geqslant 1, i \geqslant 0 \tag{2.16}$$

$$u'_{ij} = (u'_{i-1j})_X - \epsilon[P^*_X(u'_{i-1j})_X + Q^*_X(u'_{i-1j})_T] \qquad i \geqslant 1, j \geqslant 0 \tag{2.17}$$

$$u'_{00} = u + \epsilon R^* \tag{2.18}$$

to the first order in ϵ. These of course lead to considerably less cumbersome forms of u'_{ij} than those obtained from (2.10)–(2.12).

3. Properties of the transformations

Under the point transformation (2.1) each derivative of $u'(x', t')$, that is $u'_{ij}, i \geqslant 0, j \geqslant 0$, may be expressed, via the recurrence relations (2.10)–(2.12), as a function of x, t, u and the derivatives of u. A number of results concerning the functional form of $u'_{pq}(x, t, u, \ldots, u_{ij}, \ldots)$ are presented in this section. These results concern point transformations with, as yet, no reference to PDEs. In section 4, the results of section 3 are necessary to study the nature of point transformations which perform specific changes to PDEs. Of particular interest, for example, are the cases of *no* change which correspond to symmetries of the equations. The proofs of the results are generally inductive and use the recurrence relations (2.10)–(2.12). They have been relegated to appendix A.

Lemma 3.1. If $x' = P(x,t,u), t' = Q(x,t,u), u' = R(x,t,u)$

$$\sum_{i=0}^{n} z^i \frac{\partial u'_{pq}}{\partial u_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J\delta^{-p-q-1} & n > 0 \\ R_u & n = 0 \end{cases} \tag{3.1}$$

where $i + j = p + q = n \geqslant 0$.

As an illustrative example consider the point transformation

$$x' = u^{-1} \qquad t' = t \qquad u' = xu^{-1}$$

which is an element of a cyclic group of finite transformations of order 3. Direct calculation gives

$$P_X = -u^{-2}u_x \qquad P_T = -u^{-2}u_t \qquad Q_X = 0 \qquad Q_T = 1 \qquad J = u^{-3}$$
$$\delta = -u^{-2}u_x$$

and application of lemma 3.1 for $p = 0$ and $q = 2$ produces

$$z^2 \frac{\partial u'_{02}}{\partial u_{20}} + z \frac{\partial u'_{02}}{\partial u_{11}} + \frac{\partial u'_{02}}{\partial u_{02}} = -u^{-1}u_{10}^{-3}(z^2 u_{01}^2 - 2z u_{10}u_{01} + u_{10}^2)$$

using the numeric subscript notation.

Comparing coefficients of z^2, z and z^0 now leads to the following results

$$\frac{\partial u'_{t't'}}{\partial u_{xx}} = -u^{-1}u_x^{-3}u_t^2 \qquad \frac{\partial u'_{t't'}}{\partial u_{xt}} = 2u^{-1}u_x^{-2}u_t \qquad \frac{\partial u'_{t't'}}{\partial u_{tt}} = -u^{-1}u_x^{-1}.$$

These results may be readily checked against the actual expression for $u'_{t't'}$,

$$u'_{t't'} = -u^{-1}u_x^{-3}(u_t^2 u_{xx} - 2u_x u_t u_{xt} + u_x^2 u_{tt}).$$

A number of useful results are contained in lemma 3.1 and are described in the following four corollaries. First the coefficient of z^i in (3.1) gives the following.

Corollary 3.1.

$$\frac{\partial u'_{pq}}{\partial u_{ij}} = (-1)^{p+i} \sum_{s=\max\{0,p-j\}}^{\min\{i,p\}} \binom{p}{s} \binom{q}{i-s} P_X^{j-p+s} P_T^{i-s} Q_X^{p-s} Q_T^s J \delta^{-p-q-1} \tag{3.2}$$

where $i + j = p + q \geqslant 1$.

The coefficient of z^p in lemma 3.1, or setting $i = p$ and $j = q$ in corollary 3.1, gives

Corollary 3.2.

$$\frac{\partial u'_{pq}}{\partial u_{pq}} = \sum_{s=\max\{0,p-q\}}^{p} \binom{p}{s} \binom{q}{p-s} P_X^{q-p+s} P_T^{p-s} Q_X^{p-s} Q_T^s J \delta^{-p-q-1} \tag{3.3}$$

where $p + q \geqslant 1$.

The coefficient of z^n and z^0 in lemma 3.1 give, respectively:

Corollary 3.3.

$$\frac{\partial u'_{pq}}{\partial u_{p+q0}} = (-1)^q P_T^q Q_T^p J \delta^{-p-q-1} \qquad p + q \geqslant 1 \tag{3.4}$$

$$\frac{\partial u'_{pq}}{\partial u_{0p+q}} = (-1)^p P_X^q Q_X^p J \delta^{-p-q-1} \qquad p + q \geqslant 1. \tag{3.5}$$

For point transformations (2.1) restricted to $t' = Q(t)$, lemma 3.1 becomes:

Corollary 3.4. If $Q(x, t, u)$ is independent of x and u,

$$\frac{\partial u'_{pq}}{\partial u_{ij}} = \begin{cases} (-1)^{p+i} \binom{q}{i-p} P_X^{p+q-i} P_T^{i-p} Q_t^p J \delta^{-p-q-1} & i \geqslant p \\ 0 & i < p \end{cases} \tag{3.6}$$

where $i + j = p + q \geqslant 1$.

Lemma 3.2. If $x' = P(x,t,u), t' = Q(x,t,u), u' = R(x,t,u)$ then

$$\frac{\partial^{m+n}u'_{10}}{\partial u_{10}^m \partial u_{01}^n} = (-1)^n C_{mn}(n\alpha Q_X + m\beta Q_T)\delta^{-m-n-1} \tag{3.7}$$

$$\frac{\partial^{m+n}u'_{01}}{\partial u_{10}^m \partial u_{01}^n} = (-1)^m C_{mn}(n\alpha P_X + m\beta P_T)\delta^{-m-n-1} \tag{3.8}$$

where $m+n \geqslant 1$, $C_{mn} = (m+n-1)!\ \alpha^{m-1}\beta^{n-1}J$, depends only on x, t and u and where $\alpha = P_t Q_u - P_u Q_t$ and $\beta = P_x Q_u - P_u Q_x$.

When using lemma 3.2 for given values of m and n the formulae should be simplified *before* the particular forms of α and β, etc, are inserted. For example, when $n = 0$, β may be cancelled. This pre-empts a potential difficulty of dealing with β^{-1} when $\beta = 0$. We note that, when $m+n = 1$, lemma 3.2 becomes lemma 3.1, corollary 3.3 with $p+q = 1$.

As an example we consider again the cyclic transformation $x' = u^{-1}, t' = t, u' = xu^{-1}$. Thus $\alpha = u^{-2}, \beta = 0$ and $C_{mn} = (m+n-1)!u^{-2m-1}\beta^{n-1}$. We note that β has been retained in C_{mn} for cases where cancellation of β occurs. Then, for example, equation (3.8) of lemma 3.2 gives

$$\frac{\partial^{m+n}u'_{01}}{\partial u_{10}^m \partial u_{01}^n} = \begin{cases} 0 & n > 1, n+m \geqslant 1 \\ u^{-1}(-u_{10})^{-m-1} & n = 1, m \geqslant 0 \\ u^{-1}(-u_{10})^{-m-1}u_{01} & n = 0, m \geqslant 1 \end{cases}$$

which is consistent with the relation

$$u'_{t'} = -u^{-1}u_x^{-1}u_t.$$

Lemma 3.3. If $x' = P(x), t' = Q(t), u' = R(x,t,u)$ then

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial u_{kl}} = \begin{cases} \begin{pmatrix} p \\ i \end{pmatrix}\begin{pmatrix} q \\ j \end{pmatrix} P_x^{-p}Q_t^{-q}R_{uu} & i+k = p,\ j+l = q \\ 0 & i+k > p \text{ or } j+l > q. \end{cases} \tag{3.9}$$

The previous example of a cyclic transformation is not applicable here since for example, $x' = u^{-1}$ is not of the form $P(x)$. Instead we consider the reciprocal (cyclic of order 2) transformation

$$x' = x \qquad t' = t^{-1} \qquad u' = xu^{-1}.$$

Here we have

$$P_x = 1 \qquad Q_t = -t^{-2} \qquad J = xt^{-2}u^{-2} \qquad \delta = -t^{-2}$$

so that lemma 3.3 gives, when $i+k = p$ and $j+l = q$,

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial u_{kl}} = (-1)^q \begin{pmatrix} p \\ i \end{pmatrix}\begin{pmatrix} q \\ j \end{pmatrix} 2xt^{2q}u^{-3}.$$

A particular case is

$$\frac{\partial u'_{21}}{\partial u_{11}\partial u_{10}} = -4xt^2u^{-3}$$

which is consistent with the full expression for u'_{21} (or $u'_{x'x't'}$),

$$u'_{x'x't'} = t^2u^{-4}(xu^2u_{xxt} - 2xuu_tu_{xx} - 4xuu_xu_{xt} + 2u^2u_{xt} + 6xu_x^2u_t - 4uu_xu_t).$$

The above results all apply to finite transformations. However, since infinitesimal transformations have such importance in the study of PDEs, the corresponding versions of the above lemmas are given below.

For the infinitesimal transformation (2.13) lemma 3.1 condenses to

$$\frac{\partial u'_{pq}}{\partial u_{p-1q+1}} = -\epsilon p Q^*_X \qquad p \geqslant 1 \tag{3.10}$$

$$\frac{\partial u'_{pq}}{\partial u_{p+1q-1}} = -\epsilon q P^*_T \qquad q \geqslant 1 \tag{3.11}$$

$$\frac{\partial u'_{pq}}{\partial u_{pq}} = 1 - \epsilon((p+1)P^*_X + (q+1)Q^*_T - P^*_x - Q^*_t - R^*_u) \qquad p+q \geqslant 1 \tag{3.12}$$

$$\frac{\partial u'_{00}}{\partial u_{00}} = 1 + \epsilon R^*_u \tag{3.13}$$

$$\frac{\partial u'_{pq}}{\partial u_{ij}} = 0 \qquad i+j = p+q > 0 \qquad (i,j) \neq (p-1, q+1)$$
$$(i,j) \neq (p+1, q-1) \qquad (i,j) \neq (p,q) \tag{3.14}$$

to order ϵ. In the case of lemma 3.2, equation (3.7) simplifies to

$$\frac{\partial^2 u'_{10}}{\partial u^2_{10}} = -2\epsilon P^*_u \qquad \frac{\partial^2 u'_{10}}{\partial u_{10}\partial u_{01}} = -\epsilon Q^*_u \tag{3.15}$$

$$\frac{\partial u'_{10}}{\partial u_{10}} = 1 - \epsilon(2P^*_X + Q^*_T - P^*_x - Q^*_t - R^*_u) \qquad \frac{\partial u'_{10}}{\partial u_{01}} = -\epsilon Q^*_X \tag{3.16}$$

to order ϵ. For all other derivatives

$$\frac{\partial^{m+n} u'_{10}}{\partial u^m_{10}\partial u^n_{01}} = \mathrm{O}(\epsilon^2). \tag{3.17}$$

The corresponding derivatives of u'_{01} as given by (3.8) in the finite case may be obtained from (3.15)–(3.17) by exchanging x and t, X and T and P and Q. Finally the infinitesimal form of lemma 3.3 is, to order ϵ,

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial u_{kl}} = \begin{cases} \epsilon \begin{pmatrix} p \\ i \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} R^*_{uu} & i+k = p,\ j+l = p \\ 0 & i+k > p \text{ or } j+l > q. \end{cases} \tag{3.18}$$

We close this section with a conjecture concerning transformations in which the new independent variables x' and t' are defined solely in terms of x and t.

Conjecture 3.1. If $x' = P(x,t), t' = Q(x,t), u' = R(x,t,u)$ then

$$\frac{\partial^n u'_{pq}}{\partial u^{n_1}_{i_1 j_1} \ldots \partial u^{n_m}_{i_m j_m}} = (-1)^q \sum_{s=\max\{0,p-j\}}^{\theta=\min\{i,p\}} C_s P_x^{p-j+s} P_t^{i-s} Q_x^{p-s} Q_t^s \delta^{-p-q} \frac{\partial^n R}{\partial u^n} \tag{3.19}$$

where $i = \sum_{r=1}^m n_r i_r$, $j = \sum_{r=1}^m n_r j_r$, $i+j = p+q$ and where C_s is a non-zero constant depending on all of the i_r, j_r, n_r, and also on p and q.

The precise form of C_s is as yet unknown. The conjecture is supported by the fact that it agrees with lemma 3.1, corollary 3.1 for $n = 1$ and by the fact that it is valid for a number of tested cases.

4. Form-preserving transformations of PDEs

4.1. Basic results

We start with a wide class of PDEs for which general deductions about the forms of $P(x, t, u)$ and $Q(x, t, u)$ can quickly be made. These will be useful when discussing the following more restricted classes of equations. Also a general comment about the existence of hodograph-type transformations can be made. The results are summarized in the following theorem.

Theorem 4.1. The PDE $u_{pq} = H(x, t, u, \{u_{ij}\})$ is related to $u'_{pq} = H'(x', t', u', \{u'_{ij}\})$, where $\{u_{ij}\}$ and $\{u'_{ij}\}$ respectively denote all derivatives of u and u' of order $i + j < p + q$, are related by the point transformation $x' = P(x, t, u), t' = Q(x, t, u), u' = R(x, t, u)$ in the cases: (a) $p \neq 0, q \neq 0$ (b) $p \neq 0, q = 0$ (c) $p = 0, q \neq 0$ *only if* (a) $\{P = P(x), Q = Q(t)\}$ or $\{P = P(t), Q = Q(x)\}$, (b) $Q = Q(t)$, (c) $P = P(x)$, respectively.

Proof. For the proof of theorem 4.1 we consider the fate of the highest-order derivative of $u'_{pq} = H'$ under the point transformation. Consider lemma 3.1, corollary 3.3, $p + q \geqslant 1$, that is

$$\frac{\partial u'_{pq}}{\partial u_{p+q0}} = (-1)^q Q_T^p P_T^q J \delta^{-p-q-1} \tag{3.4}$$

$$\frac{\partial u'_{pq}}{\partial u_{0p+q}} = (-1)^p Q_X^p P_X^q J \delta^{-p-q-1}. \tag{3.5}$$

In case (a) neither $p = 0$ nor $q = 0$ so that both of these expressions must vanish in order for u'_{pq} to generate u_{pq} alone of order $p+q$. Any lower-order derivatives of u' which occur in H' transform to derivatives of u of order less than $p + q$. Hence, $Q_T P_T = Q_X P_X = 0$. Hence, either $\{P = P(x), Q = Q(t)\}$ or $\{P = P(t), Q = Q(x)\}$ as required.

In case (b), in which $p \neq 0$ and $q = 0$, only the expression in (3.5) must vanish, so that $Q_X = 0$ and hence $Q = Q(t)$ as required.

Case (c) follows by symmetry ($x \leftrightarrow t$, $P \leftrightarrow Q$, $X \leftrightarrow T$, $p \leftrightarrow q$) from case (b). □

A corollary of this is that for a hodograph-type transformation in which either or each of x' and t' has a dependence on u, and u' has a dependence on either or each of x, t, only cases (b) and (c) can apply. This gives a simple test for the possible existence of a hodograph-type point transformation for PDEs with two independent variables, namely:

a hodograph-type transformation can only exist if either there are at least two highest-order derivatives present, or the single highest order is pure, not mixed.

4.2. Equations of the form $u_{01} = H(x, t, u, \ldots, u_{n0})$

Spurred on by the result of Tu [11] that infinitesimal point transformations for equations of this type with $n \geqslant 2$ must take the form $t' = t + \epsilon f(t) + \mathrm{o}(\epsilon^2)$ (no x or u dependency) and also by the fact that all point transformations (discrete or continuous) connecting two different Burgers-type equations (Kingston and Sophocleous [1]) were also of this form, Kingston [12] generalized Tu's result accordingly. He also went further to show that for a wide subclass of these equations it is necessary for $x' = P(x, t)$ (no u dependency). These results are incorporated in theorems 4.2a, b.

For a subclass of these evolution equations in which H, H' have no explicit dependence on t and t' and for point transformations restricted to $x' = P(x,t), t' = Q(x,t)$. Kalnins and Miller [13] described a method of using the Lie point symmetries of one equation to obtain a point transformation connecting that equation to another. For example, they use the symmetry $t'\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} - \frac{\partial}{\partial u'}$ of the KdV equation $u'_{t'} = u'_{x'x'x'} + u'u'_{x'}$ to find the point transformation $x' = x + t^2/2, t' = t, u' = u - t$ relating the KdV equation to the new equation

$$u_t = u_{xxx} + uu_x + 1.$$

For this particular example with $H = u_{xxx} + uu_x + 1$ and $H' = u'_{x'x'x'} + u'u'_{x'}$ we note that theorems 4.2a, b predict that if P and Q exist they must be of the forms $P(x,t)$ and $Q(t)$, in accordance with the above results.

Theorem 4.2a. The point transformation $x' = P(x,t,u), t' = Q(x,t,u), u' = R(x,t,u)$ transforms

$$u'_{01} = H'(x', t', u', \ldots, u'_{n0}) \tag{4.1}$$

to

$$u_{01} = H(x, t, u, \ldots, u_{n0}) \tag{4.2}$$

where $n \geqslant 2$ if and only if $Q = Q(t)$ and

$$H = J^{-1}Q_t(P_X Q_t H' + P_t R_X - P_X R_t). \tag{4.3}$$

Proof. Theorem 4.1 applies with $p = 2$ and $q = 0$, so that $Q = Q(t)$. Each u'_{i0} in H' transforms to an expression in $x, t, u, u_{10}, \ldots u_{i0}$, that is no t derivatives of u are introduced. Equation (4.1) thus transforms to the form (4.2) and the form of H is determined, with no further conditions on P, Q and R, from (4.3) for any H'. □

Theorem 4.2b. If, in theorem 4.2a, H and H' are polynomials (non-negative integral powers) in $u_{10}, \ldots, u_{n0}$ and $u'_{10}, \ldots, u'_{n0}$ respectively (dependency on x, t, u and x', t', u' unspecified) then $P = P(x,t)$.

Proof. The proof of theorem 4.2b is given in [12].

These results have been used, for example, to aid the classification of point transformations within the following classes of PDEs: generalized Burgers equations [1]; radially symmetric nonlinear diffusion equation [14]; generalized nonlinear diffusion equations [15]. Of course these point transformations necessarily include all invariant continuous Lie point symmetries of members of each class. They also include additional discrete symmetries. For example, Pallikaros and Sophocleous [15] studied the equations

$$u_t = x^m(x^n f(u) u_x)_x$$

where m and n are any real constants. In particular the discrete reciprocal symmetries were found:

$$u_t = x(x^{-1}u^{-2}u_x)_x \qquad x' = \frac{2}{x} \qquad t' = t \qquad u' = \tfrac{1}{2}x^2 u$$

$$u_t = x(u^{-2}u_x)_x \qquad x' = \frac{1}{x} \qquad t' = t \qquad u' = xu.$$

In each of these two examples we note that the conditions of theorems 4.2a, b are met and, indeed, $x' = P(x,t)$ and $t' = Q(t)$ as predicted.

When H and H' are not of the polynomial form described in theorem 4.2b the result $x' = P(x,t)$ does not necessarily follow as can be illustrated by the *pure hodograph transformation* $x' = u$, $t' = t$, $u' = x$ in which one independent variable x and the dependent variable u change roles. Clarkson *et al* [16] used this transformation and an *extended hodograph transformation* to analyse classes of linearizable PDEs. The pure hodograph transformation leaves both $u_t = (u_{xx})^{\frac{1}{3}}$ and the third-order potential Harry–Dym equation $u_t = (u_x^{-\frac{1}{2}})_{xx}$ invariant. In each case H contains negative or fractional (or both) powers of a derivative of u, so $x' = u$ is consistent with theorem 4.2b.

Note that $x' = \frac{1}{x}$, $t' = \frac{1}{t}$, $u' = \frac{u}{x}$ leaves $u_t = (u_{xx})^{-1/3}$ invariant so the condition on H in theorem 4.2b is not a necessary condition for $x' = P(x,t)$.

4.3. Equations of the form $u_{11} = H(x, t, u, \ldots, u_{n0})$

This class of PDEs includes, for example, Liouville's equation $u_{xt} = \mathrm{e}^x$, the Tzitzeica equation $u_{xt} = \mathrm{e}^{-u} - \mathrm{e}^{-2u}$ (also known as the Dodd–Bullough equation) and the potential sine-Gordon equation $u_{xt} = u\sqrt{1-u_x^2}$.

Theorem 4.3a ($n \geqslant 3$). The point transformation $x' = P(x,t,u)$, $t' = Q(x,t,u)$, $u' = R(x,t,u)$ transforms

$$u'_{11} = H'(x', t', u', \ldots, u'_{n0}) \tag{4.4}$$

into

$$u_{11} = H(x, t, u, \ldots, u_{n0}) \tag{4.5}$$

where $n \geqslant 3$, if and only if $P = P(x,t)$, $Q = Q(t)$, $R = A(t)u + B(x,t)$ and

$$H = A^{-1}P_xQ_tH' + u_{20}P_x^{-1}P_t + u_{10}((P_x^{-1}P_t)_x - A^{-1}A_t) - A^{-1}(B_t - P_x^{-1}P_tB_x)_x. \tag{4.6}$$

Proof. From theorem 4.1 with $p = n$ and $q = 0$ it follows that $Q = Q(t)$. Relation (2.11) simplifies to $u'_{i0} = P_x^{-1}(u'_{i-10})_X$, $i \geqslant 1$, so that no t derivatives of u arise from $u'_{i0}, i \geqslant 0$, and H' transforms to the form H.

Hence, equation (4.4) only transforms to (4.5) if u'_{11} gives rise to no terms in u_{02} or u_{01}. Thus $\frac{\partial u'_{11}}{\partial u_{01}} \equiv 0$, so that $\frac{\partial}{\partial u_{01}}(\frac{\partial u'_{11}}{\partial u_{20}}) = -Q_tP_uJ\delta^{-3} \equiv 0$, using equation (3.4) of lemma 3.1, corollary 3.3, with $p = q = 1$. Hence, $P = P(x,t)$. Equation (4.4) now transforms to equation (4.5) and H is given in terms of H', whatever H', by equation (4.6). □

As an example of the use of theorem 4.3a all point transformations which are symmetries of the equation

$$u_{xt} = u_{xxx} + 2u_xu_{xx} \tag{4.7}$$

will be found. This equation may be recognized as the x-derivative of the potential Burger's equation $u_t = u_{xx} + u_x^2$ so that we can expect some Lie point symmetries to exist, at least.

Using the forms of P, Q and R in the theorem $H' = u'_{30} + 2u'_{10}u'_{20}$ may be expressed in terms of x, t, u, u_{10}, u_{20} and u_{30} and then substituted into condition (4.6). Equation (4.6) then becomes a polynomial in u_{10}, u_{20} and u_{30}. The coefficients of $u_{10}u_{20}$, u_{10}^2, u_{30} and u_{20} give respectively $A = 1$, $P_{xx} = 0$, $Q_t = P_x^2$, $B_x = -\frac{1}{2}P_x^{-1}P_t$ and the remaining terms give

$P_x P_{tt} = 2P_t P_{xt}$. The forms of P, Q and B are now easily determined and the possible symmetries of (4.7) may be represented as two classes of multiparameter transformations

$$(1) \qquad x' = \frac{c_1 x + c_3}{t + c_2} + c_4 \qquad t' = \frac{-c_1^2}{t + c_2} + c_5 \qquad u' = u + \frac{(c_1 x + c_3)^2}{4c_1^2(t + c_2)} + b(t)$$

$$(2) \qquad x' = c_1 x + c_2 t + c_3 \qquad t' = c_1^2 t + c_4 \qquad u' = u - \tfrac{1}{2} c_1^{-1} c_2 x + b(t).$$

Here $c_1, \ldots, c_5$ are arbitrary constants and $b(t)$ is an arbitrary function of t which occurs as a translation of u and is a reminder that (4.7) may be written in terms of u_x.

These two classes of transformations contain the Lie point symmetries of (4.7) and also a discrete symmetry T which is an element of a finite cyclic group of order 4 and is obtained by setting $c_1 = 1, c_2 = c_3 = c_4 = c_5 = b(t) = 0$ in class (1):

$$T: \qquad x' = xt^{-1} \qquad t' = -t^{-1} \qquad u' = u + \tfrac{1}{4} x^2 t^{-1}.$$

T^2 represents the reflection $x' = -x, t' = t, u' = u$ and T^4 is the identity transformation.

Theorem 4.3b ($n = 2$). The point transformations $x' = P(x, t, u), t' = Q(x, t, u), u' = R(x, t, u)$ which transform

$$u'_{11} = H'(x', t', u', u'_{10}, u'_{20}) \tag{4.8}$$

to

$$u_{11} = H(x, t, u, u_{10}, u_{20}) \tag{4.9}$$

belong to the two categories:

(a) P, Q, R and H restricted as in the conditions for theorem 4.3a;

(b) $P = P(x, t), Q = Q(x, t), R = A(x, t)u + B(x, t)$, $H' = -P_x Q_x^{-1} u'_{20} - A\delta^{-1}(A^{-1}Q_x^{-1}\delta)_x u'_{10} + G'(x', t', u')$, $H = Q_x^{-1} Q_t u_{20} + A^{-1}((AQ_x^{-1}Q_t)_x - A_t)u_{10} + G(x, t, u)$. For any $G'(x', t', u')$ the form of $G(x, t, u)$ is then determined by the transformation without further condition. Also, $\delta = P_x Q_t - P_t Q_x$.

The proof of theorem 4.3b is given in appendix B.

As an illustration consider the discrete (reciprocal) transformation

$$x' = x^{-1} \qquad t' = x^2 t \qquad u' = ux^{-1}.$$

This belongs to case (b) and corresponds to

$$P(x, t) = x^{-1} \qquad Q(x, t) = x^2 t \qquad A(x, t) = x^{-1} \qquad B(x, t) = 0 \qquad \delta = -1.$$

The theorem shows that the equation

$$u'_{x't'} = \tfrac{1}{2} x' t'^{-1} u'_{x'x'} + G'(x', t', u')$$

transforms into

$$u_{xt} = \tfrac{1}{2} x t^{-1} u_{xx} + G(x, t, u)$$

where G is defined for any G' by

$$G(x, t, u) = xG'(x^{-1}, x^2 t, ux^{-1}).$$

For example, the equation $u_{xt} = \frac{1}{2} x t^{-1} u_{xx} + tu^3$ is invariant under this reciprocal transformation.

Theorem 4.3c ($n = 0, 1$). The point transformations $x' = P(x, t, u), t' = Q(x, t, u), u' = R(x, t, u)$ which transform

$$u'_{11} = H'(x', t', u', u'_{10}) \tag{4.10}$$

into

$$u_{11} = H(x, t, u, u_{10}) \tag{4.11}$$

belong to one of the two categories (when $n = 0$ set A constant in (a) and (b)):

(a) $P = P(x)$, $Q = Q(t)$, $R = A(t)u + B(x, t)$

$$H = A^{-1} P_x Q_t H' - A^{-1} A_t u_{10} - A^{-1} B_{xt} \tag{4.12}$$

(b) $P = P(t)$, $Q = Q(x)$, $R = A(x, t)u + B(x, t)$, $H' = A^{-1} A_x Q_x^{-1} u'_{10} + G'(x', t', u')$, $H = -A^{-1} A_t u_{10} + A^{-1} P_t Q_x G' - u(A^{-1} A_t)_x - (A^{-1} B_t)_x$.

The proof of theorem 4.3c is deferred to appendix B.

In particular these point transformations include continuous Lie point transformations which leave equation (4.10) invariant. For example the Liouville equation $u_{xt} = \mathrm{e}^u$ possesses an infinite-dimensional symmetry which may be written in the finite form

$$x' = P(x) \qquad t' = Q(t) \qquad u' = u - \ln(P_x Q_t)$$

where $P(x)$ and $Q(t)$ are arbitrary differentiable functions and $P_x Q_t \neq 0$. This is in accordance with theorem 4.3c, case (a), with $A(t) = 1$ and $B(x, t) = -\ln(P_x Q_t)$.

4.4. Equations of the form $u_{02} = H(x, t, u, \ldots, u_{n0})$

These equations include many models of physical phenomena, especially wave-type motions. Examples include the (linear) axially symmetric wave equation $u_{tt} = u_{xx} + \frac{1}{x} u_x$, the equation $u_t = -u_x u_{xx}$ which arises as a model of steady (here t represents a spatial coordinate) transonic gas-dynamic flow, the family of nonlinear equations $u_{tt} = (f(u)u_x)_x$ and the Boussinesq-type equation

$$u_{tt} = u_{xx} - 2(u^3)_{xx} + u_{xxxx}. \tag{4.13}$$

Theorem 4.4a ($n \geqslant 3$). The point transformation $x' = P(x, t, u), t' = Q(x, t, u), u' = R(x, t, u)$ transforms

$$u'_{02} = H'(x', t', u', \ldots, u'_{n0}) \tag{4.14}$$

to

$$u_{02} = H(x, t, u, \ldots, u_{n0}) \tag{4.15}$$

where $n \geqslant 3$ if and only if $P = P(x)$, $Q = Q(t)$ and $R = A(x) Q_t^{1/2} u + B(x, t)$. Also

$$H = A(x)^{-1} Q_t^{-\frac{3}{2}} (Q_t^3 H' + Q_{tt} R_t - Q_t R_{tt}). \tag{4.16}$$

Proof. From theorem 4.1, with $p = n$ and $q = 0$, it follows that $Q = Q(t)$. Relation (2.11) simplifies to $u'_{i0} = P_X^{-1} (u'_{i-10})_X, i \geqslant 1$, so it is evident that the transformed $u'_{i0}, i \geqslant 0$, involves no t derivatives of u. Hence, (4.14) can only be transformed into (4.15) if u'_{02} does not give rise to either of the terms u_{11} or u_{01}. However, lemma 3.1, corollary 3.4 gives $\frac{\partial u'_{02}}{\partial u_{11}} = -2 P_X P_T J \delta^{-3}$, and since $P_X \neq 0$ (otherwise P and Q are functionally dependent) it follows that $P_T = 0$ so that $P = P(x)$.

Lemma 3.3 now gives $\frac{\partial^2 u'_{02}}{\partial u_{01}^2} = R_{uu} Q_t^{-2} = 0$ showing that R is linear in u. Further $\frac{\partial u'_{02}}{\partial u_{01}} = Q_t^{-3}(2 Q_t R_{ut} - Q_{tt} R_u) = 0$, so that R is necessarily of the form $R = A(x) Q_t^{1/2} u + B(x, t)$.

With these forms of P, Q and R equation (4.14) is transformed to equation (4.15) and H is given by (4.16). □

As an example, theorem 4.4a is now used to classify the point symmetries of the Boussinesq-type equation (4.13). The restricted nature of the possible point transformations in the theorem suggests that the variety of symmetries will be limited. When (4.16) is expanded the coefficients of u_{40} and u_{30} give respectively that P, Q are linear functions and A is constant. This simplifies (4.16) considerably and the uu_{20} term is then seen to imply that $B(x, t) = 0$. It then rapidly follows that all point symmetries of

$$u_{tt} = u_{xx} - 2(u^3)_{xx} + u_{xxxx}$$

are included in the transformation

$$x' = c_1 x + c_2 \qquad t' = c_3 t + c_4 \qquad u' = c_5 u$$

where c_2, c_4 are arbitrary constants corresponding to continuous translations of x and t, and $c_1^2 = c_3^2 = c_5^2 = 1$ which gives all combinations of the three discrete reflection symmetries: $x' = -x$; $t' = -t$; $u' = -u$. These reflections are, of course, clearly visible in the equation.

Theorem 4.4b ($n = 2$). Point transformations $x' = P(x, t, u), t' = Q(x, t, u)$, $u' = R(x, t, u)$ which transform

$$u'_{02} = H'(x', t', u', u'_{10}, u'_{20}) \tag{4.17}$$

into

$$u_{02} = H(x, t, u, u_{10}, u_{20}) \tag{4.18}$$

where $H'_{u'_{20}} \neq 0$, belongs to one of the three categories:

(a) P, Q, R and H restricted as in the conditions for theorem 4.4a;

(b) $P = P(t), Q = Q(x)$, $H' = H'(x', t', u', u'_{20} + \lambda u'^2_{10} + \mu u'_{10})$, where $\lambda = -R_{uu} R_u^{-2}$, $\mu = P_t^{-2} R_u^{-2}(2P_t R_t R_{uu} - 2P_t R_u R_{ut} + P_{tt} R_u^2)$, $H = H(x, t, u, u_{20} + R_{uu} R_u^{-1} u_{10}^2 + (2R_{ux} R_u^{-1} - Q_{xx} Q_x^{-1}) u_{10})$;

(c) $P = P(x, t), Q = Q(x, t), R = A(x, t)u + B(x, t)$, $H' = P_x P_t Q_x^{-1} Q_t^{-1} u'_{20} + G'_1(x', t') u'_{10} + G'_2(x', t', u')$, $H = P_x^{-1} P_t Q_x^{-1} Q_t u_{20} + G_1(x, t) u_{10} + G_2(x, t, u)$.

The proof of theorem 4.4b is given in appendix B.

The one-dimensional nonlinear wave equation

$$u_{tt} = (f(u) u_x)_x$$

where $f(u)$ is not constant, is an example of equation (4.18). The u_x^2 term which arises in this equation excludes case (c) since neither H nor H' can take this form. The theorem therefore predicts that all point transformations between such equations must be such that x' and t' take one of the two forms (a) $x' = P(x), t' = Q(t)$, (b) $x' = P(t), t' = Q(x)$. Ames *et al* [17] classified the Lie point symmetries of this class of equations and found that, depending on the nature of $f(u)$, the symmetry Lie algebra is 3, 4 or 5 dimensional. All of these are of type (a). As well as these continuous symmetries there is a discrete symmetry for general $f(u)$ in which x and t change roles (see for example [18]). This is of type (b). For further study, the theorem could be used to classify all point symmetries including all discrete symmetries as well as the known continuous symmetries.

Another example is provided by the symmetry $2xt\frac{\partial}{\partial x} + (x^2 + t^2)\frac{\partial}{\partial t} - ut\frac{\partial}{\partial u}$ of the axially symmetric wave equation $u_{tt} = u_{xx} + \frac{1}{x} u_x$. The finite form of this is $x' = c(x, t)^{-1} x, t' =$

$c(x,t)^{-1}(t+\alpha(x^2-t^2))$, $u' = c(x,t)^{\frac{1}{2}}u$ where $c(x,t) = 1-2\alpha t-\alpha^2(x^2-t^2)$. This one-parameter family of point transformations belongs to the class described in theorem 4.4b, case (c), in which $A(x,t) = c(x,t)^{\frac{1}{2}}$ and $B(x,t) = 0$.

We know of no studies of the full (discrete as well as continuous) point symmetry groups of equations or families of equations of the type of section 4.4.

Theorem 4.4c ($n = 1$). Point transformations $x' = P(x,t,u)$, $t' = Q(x,t,u)$, $u' = R(x,t,u)$ which transform

$$u'_{02} = H'(x', t', u', u'_{10}) \tag{4.19}$$

into

$$u_{02} = H(x,t,u,u_{10}) \tag{4.20}$$

where $H'_{u'_{10}} \neq 0$, belong to one of the three categories:

(a) P, Q, R and H restricted as in the conditions for theorem 4.4a;

(b) $P = P(x)$, $Q = Q(x,t)$, $Q_x \neq 0$, $R = A(x,t)u + B(x,t)$, $H' = G'_1(x',t')u'_{10} + G'_2(x',t',u')$, $H = G_1(x,t)u_{10} + G_2(x,t,u)$, where

$$G'_1 = P_x Q_x^{-1} Q_t^{-1}(Q_t^{-1}Q_{tt} - 2A^{-1}A_t)$$
$$G_1 = Q_x^{-1}Q_t(Q_t^{-1}Q_{tt} - 2A^{-1}A_t)$$
$$G_2 = A^{-1}Q_t^2 G'_2 + A^{-2}Q_x^{-1}(F_1 u + F_2)$$
$$F_1 = Q_{tt}AA_x - 2Q_t A_x A_t + Q_x(2A_t^2 - AA_{tt})$$
$$F_2 = Q_{tt}AB_x - 2Q_t B_x A_t + Q_x(2A_t B_t - AB_{tt})$$

(c) $P = P(x)$, $Q_u \neq 0$, $Q = f(\xi)$, $R = g(\xi) + A(x,t)f'(\xi)^{1/2}$, $\xi = A_t^{1/2}u + B(x,t)$, $H' = G'_1(u'_{10} + G')^3 + G'_2$, $H = G_1(u_{10} + G)^3 + G_2$, where

$$G'_1 = A_t^{-3}P_x^3 f'(\xi)^{-3}G_1$$
$$G'_2 = f'(\xi)^{-1}((f'(\xi)^{-\frac{1}{2}})''A + (g'(\xi)f'(\xi)^{-1})')$$
$$G_2 = -A_t^{\frac{1}{2}}((\tfrac{1}{2}A_t^{-\frac{3}{2}}A_{tt})_t u + (B_t A_t^{-1})_t)$$
$$G' = A_x P_x^{-1} f'(\xi)$$
$$G = \tfrac{1}{2}A_t^{-1}A_{xt}u + A_t^{-\frac{1}{2}}B_x.$$

In the above A, B, f and g are arbitrary (suitably differentiable) functions.

The proof of theorem 4.4c is deferred to appendix B.

The class of equations (4.20) considered in theorem 4.4c is done so for completeness of the study in section 4.4. Perhaps a more natural form of (4.20) is, switching x and t, the class of evolution equations

$$u_{01} = F(x,t,u,u_{20})$$

which is a subclass of the equations studied in section 4.2. However, (4.20) is developed further here.

In all three possible cases it is noted that x' is of the form $P(x)$. The form of t' changes from $Q(t)$ in (a) to $Q(x,t)$ in (b) and finally to $Q(x,t,u)$ in (c). The latter case offers the possibility of hodograph-type transformations in which there is some interchange between dependent and independent variables. The nature of the function $H(x,t,u,u_{10})$ differs in the three cases. In (a) the form of H can be very varied whereas in (b) H has to be linear in u_{10}. Case (c) is perhaps the most interesting since as well as a more interesting transformation the functions H and H' are cubic functions of u_{10} and u'_{10} respectively.

An example is therefore given of case (c) by selecting

$$P(x) = x \qquad f(\xi) = \xi \qquad g(\xi) = 0 \qquad A(x,t) = -t^{-1} \qquad B(x,t) = 0$$

which gives

$$G = G' = G_2 = G'_2 = 0 \qquad Q = ut^{-1}, R = -t^{-1}$$

and implies that $u'_{t't'} = G'_1 u'^3_{x'}$ transforms into $u_{tt} = G_1 u_x^3$ where $G_1(x,t,u) = t^{-6} G'_1(x, ut^{-1}, -t^{-1})$

In particular, choosing $G'_1(x', t', u') = (t'u')^{-2}$ gives $G_1(x,t,u) = (tu)^{-2}$. Thus, the result follows that the equation

$$u_{tt} = (tu)^{-2} u_x^3$$

has the discrete symmetry

$$x' = x \qquad t' = ut^{-1} \qquad u' = -t^{-1}.$$

This transformation belongs to a cyclic group of order 3.

5. Conclusion

We have drawn attention to the fact that in addition to the important Lie symmetry groups of PDEs there may also be discrete finite groups of point transformations which, with the Lie groups, form the complete symmetry group under point transformations. In the introduction some reasons were listed for why it may be rewarding to search for discrete point transformations. To this end we have studied three common classes of equations restricted to one dependent variable and two independent variables and deduced results, summarized in theorems, concerning the nature of connecting point transformations.

At present there appear to be very few complete analyses of full symmetry groups (discrete and continuous) of PDEs, unlike the Lie symmetry groups for which a large catalogue (see, for example, Ibragimov [18]) exists. It is relatively easy to construct PDEs artificially with given finite symmetry groups. For example, $u_t = (x^2 u u_{xx})^{1/3}$ is invariant under $x' = 1/u, t' = t, u' = x/u$, which transformation forms a cyclic group of order 3. What variety of finite symmetry groups exist within PDEs of practical importance is largely an open question.

For systems of equations with two or more independent variables the situation is the same regarding the complete knowledge of the full group of invariant transformations, discrete transformations included. The self-consistent system of two equations

$$\Psi_{xxt} = \Psi_x \Psi_{xt}/\Psi + (\Psi_{xx}\Psi_{xt} + \lambda \Psi \Psi_t)/\Psi_x$$
$$\Psi_{xtt} = \Psi_t \Psi_{xt}/\Psi + (\Psi_{tt}\Psi_{xt} + \lambda^{-1} \Psi \Psi_x)/\Psi_t$$

is invariant under the point transformation (reciprocal) $\Psi \to 1/\Psi, x \to -x, t \to -t$. These two equations arise from the linear representation of the Dodd–Bullough equation $u_{xt} = \mathrm{e}^u - \mathrm{e}^{-2u}$, namely

$$\Psi_{xx} = u_x \Psi_x - \lambda \mathrm{e}^{-u} \Psi_t \qquad \Psi_{xt} = \mathrm{e}^u \Psi \qquad \Psi_{tt} = -\lambda^{-1} \mathrm{e}^{-u} \Psi_t + u_t \Psi_t$$

Ψ being the eigenfunction.

Also there are useful point transformations between different equations with more than two independent variables. For example the Kadomtsev–Petvialishvili (KP) equation

$$(u'_{t'} + 6u'u'_{x'} + u'_{x'x'x'})_{x'} = -3\alpha^2 u'_{y'y'}$$

and Johnson's equation [19]

$$\left(u_t + 6uu_x + u_{xxx} + \frac{u}{2t}\right)_x = -3\alpha^2 u_{yy}/t^2$$

are connected by the point transformation

$$x' = x - yt/(12\alpha^2) \qquad y' = yt \qquad t' = t \qquad u' = u.$$

This enables solutions of Johnson's equation to be constructed from solutions of the K-P equation.

Further study, along the lines of this paper, of systems of equations or a single equation with more than two independent variables may therefore be useful.

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Appendix A

In this appendix, we give the proofs of lemmas 3.1–3.3.

Proof of lemma 3.1. First note that u'_{pq} may be expressed as a function of x, t, u and the derivatives of u up to order $p + q$. Hence, both $(u'_{pq})_X$ and $(u'_{pq})_T$ will be linear in $(p + q + 1)$th-order derivatives of u. Specifically, if $i + j = p + q + 1$,

$$\frac{\partial (u'_{pq})_X}{\partial u_{ij}} = \begin{cases} \dfrac{\partial' u_{pq}}{\partial u_{i-1j}} & i \geqslant 1 \\ 0 & i = 0 \end{cases} \tag{A.1}$$

$$\frac{\partial (u'_{pq})_T}{\partial u_{ij}} = \begin{cases} \dfrac{\partial' u_{pq}}{\partial u_{ij-1}} & j \geqslant 1 \\ 0 & j = 0. \end{cases} \tag{A.2}$$

We use induction on n. For $p \geqslant 1$ and $i + j = p + q = n + 1, n \geqslant 1$, relation (2.11) gives

$$\sum_{i=0}^{n+1} z^i \frac{\partial u'_{pq}}{\partial u_{ij}} = \sum_{i=0}^{n+1} z^i \frac{\partial}{\partial u_{ij}} (\delta^{-1}(Q_T (u'_{p-1q})_X - Q_X (u'_{p-1q})_T))$$

$$= \delta^{-1} \left(\sum_{i=1}^{n+1} z^i Q_T \frac{\partial u'_{p-1q}}{\partial u_{i-1j}} - \sum_{i=0}^{n} z^i Q_X \frac{\partial u'_{i-1j}}{\partial u_{ij-1}} \right)$$

using (A.1) and (A.2) and noting that $i + j \geqslant 2$ and that δ, Q_X and Q_T involve only first derivatives of u,

$$= -\delta^{-1}(Q_X - zQ_T) \sum_{i=0}^{n} z^i \frac{\partial u'_{p-1q}}{\partial u_{ij-1}}$$

noting that j does not attain 0,

$$= (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J \delta^{-p-q-1} \tag{A.3}$$

using the induction hypothesis. Equation (A.3) is lemma 3.1 with $n+1$ in place of n.

For $q \geqslant 1$ the corresponding steps, using (2.10) initially, gives exactly the same result (A.3).

For the basis of the induction we need to consider the two cases for $n = 1$, namely $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$, and the case $n = 0$. When $(p, q) = (1, 0)$ it is necessary to show that

$$\frac{\partial u'_{10}}{\partial u_{01}} + z\frac{\partial u'_{10}}{\partial u_{10}} = -(Q_X - zQ_T)J\delta^{-2}$$

and it may be readily verified by direct calculations that

$$\frac{\partial u'_{10}}{\partial u_{01}} = -Q_X J\delta^{-2} \qquad \frac{\partial u'_{10}}{\partial u_{10}} = Q_T J\delta^{-2}.$$

The case $(p, q) = (0, 1)$ follows similarly. Finally, the basis is completed by the case $n = 0$, for which

$$\sum_{i=0}^{n} z^i \frac{\partial u'_{pq}}{\partial u_{ij}} = \frac{\partial u'_{00}}{\partial u_{00}} = R_u$$

as required. □

Proof of lemma 3.2. To prove (3.7) we use induction on $m+n$. Assuming that (3.7) is valid for a given $m+n \geqslant 1$,

$$\frac{\partial^{m+n+1}u'_{10}}{\partial u_{10}^{m+1}\partial u_{01}^{n}} = (-1)^n C_{m,n}(n\alpha Q_u\delta + (m+n+1)\alpha(n\alpha Q_X + m\beta Q_T))\delta^{-m-n-2}$$
$$= (-1)^n C_{m,n}\alpha(m+n)(n\alpha Q_X + (m+1)\beta Q_T))\delta^{-m-n-2}$$

using the fact that $\alpha Q_x - \beta Q_t + \gamma Q_u = \frac{\partial(Q,P,Q)}{\partial(x,t,u)} = 0$,

$$= (-1)^n C_{m+1,n}(n\alpha Q_X + (m+1)\beta Q_T)\delta^{-m-n-2}.$$

Similarly it may be shown that

$$\frac{\partial^{m+n+1}u'_{10}}{\partial u_{10}^{m}\partial u_{01}^{n+1}} = (-1)^n C_{m,n+1}((n+1)\alpha Q_X + m\beta Q_T)\delta^{-m-n-2}.$$

The basis of the induction consists of the two cases, $(m, n) = (1, 0)$ and $(m, n) = (0, 1)$, for which $m+n = 1$. Lemma 3.1, corollary 3.3, or direct checking, shows that

$$\frac{\partial u'_{10}}{\partial u_{10}} = Q_T J\delta^{-2} \qquad \frac{\partial u'_{10}}{\partial u_{01}} = -Q_X J\delta^{-2}$$

which conform to (3.7) as required.

Equation (3.8) follows from (3.7) by symmetry. Switch x and t, X and T, P and Q, α and β, and m and n, but note that, for these changes, $C_{m,n}$ remains as $C_{m,n}$. □

Proof of lemma 3.3. Induction may be used on $p+q = n$. Suppose (3.9) is true for all p, q such that $p+q \leqslant n-1$ for a given n and consider $p+q = n$.

For $p \geqslant 1$ (and $q \geqslant 0$, $i \geqslant 0$, $j \geqslant 0$),

$$\frac{\partial u'_{pq}}{\partial u_{ij}} = P_x^{-1}\frac{\partial}{\partial u_{ij}}\left\{(u'_{p-1q})_x + \sum_{\alpha,\beta} u_{\alpha+1\beta}\frac{\partial u'_{p-1q}}{\partial u_{\alpha\beta}}\right\}$$

using (2.11), which reduces to $u'_{pq} = P_x^{-1}(u'_{p-1q})_X$, and (2.5),

$$= P_x^{-1}\left\{\left(\frac{\partial u'_{p-1q}}{\partial u_{ij}}\right)_x + \sum_{\alpha,\beta} u_{\alpha+1\beta}\frac{\partial^2 u'_{p-1q}}{\partial u_{ij}\partial u_{\alpha\beta}} + \lambda\right\}$$

where $\lambda = \frac{\partial u'_{p-1q}}{\partial u_{i-1j}}$ if $i \geqslant 1$; $\lambda = 0$ if $i = 0$.

Hence,

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial u_{kl}} = P_x^{-1}\left\{\xi_x + \mu + \sum_{\alpha,\beta} u_{\alpha+1\beta}\frac{\partial \xi}{\partial u_{\alpha\beta}} + \nu\right\}$$

where

$$\xi = \frac{\partial^2 u'_{p-1q}}{\partial u_{ij}\partial u_{kl}}$$

$$\mu = \frac{\partial^2 u'_{p-1q}}{\partial u_{ij}\partial u_{k-1l}} \qquad \text{if } k \geqslant 1 \qquad \mu = 0 \qquad \text{if } k = 0$$

$$\nu = \frac{\partial \lambda}{\partial u_{kl}} = \frac{\partial^2 u'_{p-1q}}{\partial u_{i-1j}\partial u_{kl}} \qquad \text{if } i \geqslant 1 \qquad \nu = 0 \qquad \text{if } i = 0.$$

For $i + k = p$ and $j + l = q$ the induction hypothesis gives $\xi = 0$ since $i + k > p - 1$. Also, using the expressions for μ and ν given by (3.9) under the induction hypothesis gives $\xi = \mu = \nu = 0$.

A symmetrical argument may be used when $q \geqslant 1$ and $p \geqslant 0$ by using (2.6) in place of (2.5) at the start of the above calculations.

Hence, a basis of the induction is provided by $n = 0$. Here $p = q = 0$ and (3.9) is evidently true. □

Appendix B

In this appendix, we give the proofs of theorems 4.3b, 4.3c, 4.4b and 4.4c.

Proof of theorem 4.3b. Let $E = u'_{11} - H'$, apply the transformation and then substitute $u_{11} = H$. E will now, possibly, depend on $x, t, u, u_{10}, u_{01}, u_{02}$ and u_{20}, but for equation (4.8) to transform into equation (4.9) we require that $E \equiv 0$.

In particular,

$$\frac{\partial E}{\partial u_{02}} = \frac{\partial u'_{11}}{\partial u_{02}} - \frac{\partial H'}{\partial u'_{20}}\frac{\partial u'_{20}}{\partial u_{02}} \equiv 0$$

and from the two cases of lemma 3.1, corollary 3.3, equation (3.5), corresponding to $p = q = 1$ and $p = 2, q = 0$, we have

$$-Q_X J\delta^{-3}(P_X + Q_X u'_{20}) \equiv 0.$$

Hence, either (a) $Q_X = 0$, so that $Q = Q(t)$, or (b) $Q_X \neq 0, H' = -P_X Q_X^{-1} u'_{20} + G'_2(x', t', u', u'_{10})$.

For case (a) the same analysis applies as for theorem 4.3a.

For case (b) equation (4.8) is linear in the second-order derivatives of u' and this will transform into an equation which is also linear in second-order derivatives. Thus

$$H = G_1(x, t, u, u_{10})u_{20} + G_2(x, t, u, u_{10}).$$

Since $\frac{\partial E}{\partial u_{20}} = -\delta^{-1}Q_x^{-1}R_u(G_1Q_x - Q_t) \equiv 0$, it follows that $G_1 = Q_x^{-1}Q_t$. Next, $\frac{\partial^2 E}{\partial u_{01}^2} = -\delta^{-2}G'_{2u'_{10}u'_{10}}$ and $\frac{\partial^2 E}{\partial u_{01}\partial u_{10}} = -\delta^{-1}R_{uu}$ so that

$$G'_2 = G'_3(x', t', u')u'_{10} + G'(x', t', u')$$

and

$$R = A(x, t)u + B(x, t).$$

Also $\frac{\partial^2 E}{\partial u_{10}^2} = -\delta^{-1}AG_{2u_{10}u_{10}} \equiv 0$, giving

$$G_2 = G_3(x, t, u)u_{10} + G(x, t, u).$$

Solving $\frac{\partial E}{\partial u_{10}} \equiv 0$ and $\frac{\partial E}{\partial u_{01}} \equiv 0$ simultaneously gives

$$G_3 = A^{-1}((AQ_x^{-1}Q_t)_x - A_t) \qquad G'_3 = -A\delta^{-1}(A^{-1}Q_x^{-1}\delta)_x.$$

Finally, $E \equiv 0$ provides a lengthy relation between G and G' which serves to determine $G(x, t, u)$ corresponding to any $G'(x', t', u')$. □

Proof of theorem 4.3c. From theorem 4.1 with $p = q = 1$ we have two cases to consider: (a) $P = P(x), Q = Q(t)$; (b) $P = P(t), Q = Q(x)$. These are distinct cases since equation (4.10) is not symmetric in x' and t'.

For case (a) H' transforms into a function of x, t, u and u_{10} so we require that u'_{11} transforms into a function of the same variables, having replaced u_{11} by H. Hence, $\frac{\partial u'_{11}}{\partial u_{01}} = \delta^{-1}(R_{uu}u_{10} + R_{ux}) \equiv 0$, giving

$$R = A(t)u + B(x, t).$$

Equation (4.10) now transforms into (4.11) with H as stated in (4.12).

In case (b) let $E = u'_{11} - H'$ with H substituted for u_{11}. Thus, $E \equiv 0$ for the given transformation to exist. Hence, $\frac{\partial E}{\partial u_{10}} = \delta^{-1}(R_{uu}u_{01} + R_{ut} + R_u H_{u_{10}}) \equiv 0$, giving

$$R = A(x, t)u + B(x, t)$$

and

$$H = -A^{-1}A_t u_{10} + G(x, t, u).$$

Also $\frac{\partial E}{\partial u_{01}} = \delta^{-1}(A_x - AQ_x H'_{u'_{10}}) \equiv 0$, so that

$$H' = A^{-1}A_x Q_x^{-1}u'_{10} + G'(x', t', u').$$

Equation (4.10) now transforms into (4.11) with G being determined by G' as

$$G(x, t, u) = A^{-1}P_t Q_x G' - u(A^{-1}A_t)_x - (A^{-1}B_t)_x$$

and the proof of case (b) and theorem 4.3c is complete for $n = 1$. When $n = 0$, H and H' contain no derivatives of u and u' respectively and the further restriction $A =$ constant must apply. □

Proof of theorem 4.4b. The expression $E = u'_{02} - H'$ becomes, as a result of the point transformation, an expression in x, t, u and the derivatives of u up to order 2. This expression ($= 0$) is to be identified with equation (4.18). That is, if u_{02} is replaced by H in E then the resulting expression is required to be identically zero in terms of the remaining seven variables $x, t, u, u_{10}, u_{01}, u_{20}$ and u_{11}.

In particular, $\frac{\partial E}{\partial u_{11}} = 0$ and $\frac{\partial E}{\partial u_{20}} = 0$, using the corollaries of lemma 3.1, give

$$Q_X Q_T H'_{u'_{20}} = P_X P_T$$

and

$$H'_{u'_{20}}(Q_T^2 + Q_X^2 H_{u_{20}}) = P_T^2 + P_X^2 H_{u_{20}}.$$

These conditions show that all possibilities are included in the three cases: (a) $P = P(x)$, $Q = Q(t)$; (b) $P = P(t)$, $Q = Q(x)$; and (c)

$$H' = P_X P_T Q_X^{-1} Q_T^{-1} u'_{20} + G'(x', t', u', u'_{10}) \tag{B.1}$$

$$H = P_X^{-1} P_T Q_X^{-1} Q_T u_{20} + G(x, t, u, u_{10}). \tag{B.2}$$

Case (a) follows exactly as in the proof of theorem 4.4a following the stage at which $P = P(x)$ and the results here are exactly as in the sole case of that theorem.

In case (b) u'_{02} transforms to an expression in x, t, u, u_{10} and u_{20}, and thus contributes to the right-hand side of equation (B.1). However, $H'(x', t', u', u'_{10}, u'_{20})$ introduces u_{01} via u'_{10} and both u_{01} and u_{02} via u'_{20}. To obtain equation (4.18) it is therefore necessary that u_{01} cancels out between the fourth and fifth parameters of H'. Noting that

$$u'_{01} = (u_{01} R_u + R_t) P_t^{-1}$$

$$u'_{02} = (u_{02} P_t R_u + u_{01}^2 P_t R_{uu} + u_{01}(2P_t R_{tu} - P_{tt} R_u) + P_t R_{tt} - P_{tt} R_t) P_t^{-3}$$

it is straightforward to show that H' must take the functional form claimed in the theorem, case (b). The term u_{01} now disappears from H' and the transform of $u'_{02} = H'$ allows u_{02} to be expressed as a function of x, t, u and u'_{02}, the form of H claimed in the theorem.

Finally in case (c) we have H' and H given by (B.1) and (B.2). H is independent of u_{01}, so that (B.2) implies that $P_X^{-1} P_T Q_X^{-1} Q_T$ is independent of u_{01}. It readily follows that $P = P(x, t)$ and $Q = Q(x, t)$. Considering again $E = u'_{02} - H'$, transformed, with u_{02} replaced by H, we have successively

$$\frac{\partial^2 E}{\partial u_{10} \partial u_{01}} = -Q_x Q_t R_u^2 G'_{u'_{10} u'_{10}} \delta^{-2} = 0$$

giving

$$G' = G'_1(x', t', u') u'_{10} + G'_2(x', t', u') \tag{B.3}$$

$$\frac{\partial^2 E}{\partial u_{01}^2} = 2 P_x Q_t^{-1} R_{uu} \delta^{-1} = 0$$

giving

$$R = A(x, t) u + B(x, t) \tag{B.4}$$

and

$$\frac{\partial^2 E}{\partial u_{10}^2} = P_x Q_t^{-1} R_u G_{u_{10} u_{10}} \delta^{-1} = 0$$

giving

$$G = G_1(x, t, u) u_{10} + G_2(x, t, u). \tag{B.5}$$

Finally $\frac{\partial^2 E}{\partial u_{01} \partial u} = 0$ and $\frac{\partial^2 E}{\partial u_{10} \partial u} = 0$ give $G'_{1u'} = 0$ and $G_{1u} = 0$ which, combined with (B.3)–(B.5), completes the proof of case (c) of the theorem. □

Proof of theorem 4.4c. Theorem 4.1 with $p = 0$ and $q = 2$ gives $P = P(x)$. As earlier we set $E = u'_{02} - H'$, use the point transformation and then replace u_{02} by H. Thus E can be expressed in terms of x, t, u, u_{10} and u_{01} which expression must identically vanish. We consider derivatives of $E_1 = E\delta^3$. In particular, $\frac{\partial^2 E_1}{\partial u_{01}\partial u_{10}} = 0$ implies that

$$2Q_u H'_{u'_{10}} - JP_x^{-2} Q_X Q_T^{-1} H'_{u'_{10}u'_{10}} = 0. \tag{B.6}$$

This enables the analysis to be separated into the three cases: (a) $Q = Q(t)$; (b) $Q = Q(x, t)$, $Q_x \neq 0$, $H' = G'_1(x', t', u')u'_{10} + G'_2(x', t', u')$; (c) $Q_u \neq 0$.

Case (a) may be completed by noting that

$$\frac{\partial E_1}{\partial u_{01}} = P_x^3(2u_{01}Q_t R_{uu} + 2Q_t R_{ut} - Q_{tt} R_u) = 0$$

which gives $R_{uu} = 0$ and $2Q_t R_{ut} = Q_{tt} R_u$. These give the form of $R(x, t, u)$ as claimed in the theorem. $E_1 = 0$ then shows how H is related to H' as displayed in theorem 4.4c (a), the same as for theorem 4.4a and theorem 4.4b case (a).

In case (b) the identities $\frac{\partial^2 E_1}{\partial u_{10}^2} = 0$ and $\frac{\partial^2 E_1}{\partial u_{01}^2} = 0$ give respectively,

$$H = G_1(x, t, u)u_{10} + G_2(x, t, u)$$

and

$$R = A(x, t)u + B(x, t).$$

Then $\frac{\partial E_1}{\partial u_{10}} = 0$ and $\frac{\partial E_1}{\partial u_{01}} = 0$, together, lead to the forms of G'_1 and G_1 claimed in the theorem. E_1 is now identically zero provided that G_2 and G'_2 are related as stated.

For the third case (c) we consider equation (B.6) and express u_{01} in terms of u'_{01}; u_{10} in terms of u'_{10} and u'_{01}. This gives

$$(u'_{10} + G')H'_{u'_{10}u'_{10}} - 2H'_{u'_{10}} = 0 \tag{B.7}$$

where

$$G' = P_x^{-1} Q_u^{-1}(Q_x R_u - Q_u R_x). \tag{B.8}$$

Equation (B.7) integrates to give

$$H' = G'_1(x', t', u')(u'_{10} + G')^3 + G'_2(x', t', u'). \tag{B.9}$$

Consideration of $\frac{\partial^4 E_1}{\partial u_{10}^4} = 0$ shows that $H(x, t, u, u_{10})$ is a cubic in u_{10}, that is

$$H = G_1(x, t, u)u_{10}^3 + F_1(x, t, u)u_{10}^2 + F_2(x, t, u)u_{10} + F_3(x, t, u). \tag{B.10}$$

E_1 is now the sum of a cubic in u_{10} and a cubic in u_{01}. For convenience we will denote the coefficient of $u_{10}^m u_{01}^n$ in E_1 by $E_1[m, n]$. The identities $E_1[2, 0] = 0$ and $E_1[1, 0] = 0$ give respectively $F_1 = 3G_1 Q_x Q_u^{-1}$ and $F_2 = 3G_1 Q_x^2 Q_u^{-2}$, which from (B.8) enables H to be written as

$$H = G_1(x, t, u)(u_{10} + G)^3 + G_2(x, t, u) \tag{B.11}$$

where

$$G = Q_x Q_u^{-1}. \tag{B.12}$$

Also $E_1[0, 3] = 0$, $E_1[3, 0] = 0$ and $E_1[0, 0] = 0$ give respectively

$$G'_2 = (Q_u R_{uu} - R_u Q_{uu})Q_u^{-3} \tag{B.13}$$

$$G'_1 = G_1 P_x^5 J^{-2} \tag{B.14}$$

$$G_2 = P_x^{-1} Q_u^{-3} J(Q_u^3(Q_t R_{tt} - Q_{tt} R_t) - Q_t^3(Q_u R_{uu} - Q_{uu} R_u)). \tag{B.15}$$

Finally we encounter two restrictions on the forms of $Q(x,t,u)$ and $R(x,t,u)$ implied by $E_1[0,2]=0$ and $E_1[0,1]=0$. These equations may both be integrated with respect to u and lead to

$$Q_tR_u - Q_uR_t = (A_t^{\frac{1}{2}}Q_u)^{\frac{3}{2}} \tag{B.16}$$

$$Q_tQ_u^{-1} = \tfrac{1}{2}A_t^{-1}A_{tt}u + A_t^{-\frac{1}{2}}B_t \tag{B.17}$$

where $A(x,t)$ and $B(x,t)$ are arbitrary functions (suitably differentiable). The contrived form of the appearance of A and B in (B.16) and (B.17) is so that the general solution of (B.16) and (B.17) takes the simple form

$$Q = f(\xi) \qquad R = g(\xi) - Af'(\xi)^{\frac{1}{2}} \tag{B.18}$$

where

$$\xi = A_t^{\frac{1}{2}}u + B \tag{B.19}$$

and f and g are general.

Incorporating these forms of Q and R into the expressions for G', G_1', G_2', G, G_2 given by (B.8), (B.14), (B.13), (B.12) and (B.15) respectively leads directly to the conditions stated in theorem 4.4c (c). Under these conditions E_1 is identically zero and the stated transformation is achieved. □

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